

ON MAXIMUM SIGNLESS LAPLACIAN ESTRADA INDEX OF GRAPHS WITH GIVEN PARAMETERS II

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Recently Ayyaswamy [1] have introduced a novel concept of the signless Laplacian Estrada index (after here SLEE) associated with a graph G . After works, we have identified the unique graph with maximum SLEE with a given parameter such as: number of cut edges, pendent vertices, (vertex) connectivity and edge connectivity. In this paper we continue our characterization for two further parameters; diameter and number of cut vertices.

Keywords: Estrada index, signless Laplacian Estrada index, extremal graph, diameter, cut vertex.

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1. Introduction

Let $G = (V, E)$ be a simple, finite, and undirected graph with vertex set $V(G)$ and the edge set $E(G)$ and $|V(G)| = n$. The adjacency matrix $A = A(G) = [a_{ij}]$ of G is the binary matrix, where the element a_{ij} is equal to 1 if vertices i and j are adjacent, and 0 otherwise. The matrix $L = D - A$, where $D = \text{diag}(d_1, d_2, \dots, d_n)$ is the diagonal matrix of vertex degrees, is known as the Laplacian matrix of G . The matrix $Q = D + A$ is called the signless Laplacian matrix of G . We denote the spectrum of A , L and Q by $(\lambda_1, \lambda_2, \dots, \lambda_n)$, $(\mu_1, \mu_2, \dots, \mu_n)$ and (q_1, q_2, \dots, q_n) , respectively. For a graph G , Estrada [9] has defined the *Estrada index* of G as

$$EE(G) = \sum_{i=1}^n e^{\lambda_i}.$$

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Fath-Tabar et al. [16] proposed the *Laplacian Estrada index*, in full analogy with Estrada index as

$$LEE(G) = \sum_{i=1}^n e^{\mu_i}.$$

Theories of Estrada and Laplacian Estrada indices of graphs have been extensively studied by several authors (see [2,4-7,9-16,19-26]).

Recently, Ayyaswamy [1] developed the innovative notion of the *signless Laplacian Estrada index* as

$$SLEE(G) = \sum_{i=1}^n e^{q_i}.$$

He also established lower and upper bounds for $SLEE$ in terms of the number of vertices and edges. Grone and Merris [17, 18] proved that for a bipartite graph G , $SLEE(G) = LEE(G)$.

Previously in [8], we characterized the unique graphs with maximum $SLEE$ among the set of all graphs with given number of cut edges, pendent vertices, (vertex) connectivity and edge connectivity. In this paper, we continue our research by characterizing the unique graph according to two further parameters: diameter and number of cut vertices.

2. Preliminaries and Lemmas

In this section, we first introduce basic definitions, notations and concepts used thorough this paper and restate some proved results found in [3, 8]. Then, we prove some needful propositions for proving the main result of the next section.

definition 2.1. [3] A *semi-edge walk* of length k in graph G , is an alternating sequence $W = v_1 e_1 v_2 e_2 \dots v_k e_k v_{k+1}$, where $v_1, v_2, \dots, v_k, v_{k+1} \in V(G)$, and $e_1, e_2, \dots, e_k \in E(G)$ such that the vertices v_i and v_{i+1} are (not necessarily distinct) end points of edge e_i , for any $i = 1, 2, \dots, k$. If $v_1 = v_{k+1}$, then we say W is a *closed semi-edge walk*.

By following [8], we denote The k -th signless Laplacian spectral moment of the graph G by $T_k(G)$, i.e., $T_k(G) = \sum_{i=1}^n q_i^k$.

Theorem 2.2. [3] For a graph G , the signless Laplacian spectral moment T_k is equal to the number of closed semi-edge walks of length k .

Let G and G' be two graphs, and $x, y \in V(G)$, and $x', y' \in V(G')$. Denote by $SW_k(G; x, y)$ the set of all semi-edge walks of length k in graph G , which are beginning at vertex x , and ending at vertex y . For convenience, we use $SW_k(G; x, x)$ instead of $SW_k(G; x)$, and set $SW_k(G) = \bigcup_{x \in V(G)} SW_k(G; x)$. Thus, by Theorem 2.2, we have $T_k(G) = |SW_k(G)|$. Note that, by Taylor

expansions, we have

$$SLEE(G) = \sum_{k \geq 0} \frac{T_k(G)}{k!}.$$

By $(G; x, y) \preceq_s (G'; x', y')$ we mean $|SW_k(G; x, y)| \leq |SW_k(G'; x', y')|$, for any $k \geq 0$. Moreover, if $(G; x, y) \preceq_s (G'; x', y')$, and there exists some k_0 such that $|SW_{k_0}(G; x, y)| < |SW_{k_0}(G'; x', y')|$, then we write $(G; x, y) \prec_s (G'; x', y')$.

Lemma 2.3. [8] Let G be a graph. If an edge e does not belong to $E(G)$, Then $SLEE(G) < SLEE(G + e)$.

Lemma 2.4. [8] Let G be a graph and $v, u, w_1, w_2, \dots, w_r \in V(G)$. suppose that $E_v = \{e_1 = vw_1, \dots, e_r = vw_r\}$ and $E_u = \{e'_1 = uw_1, \dots, e'_r = uw_r\}$ are subsets of edges of the complement of G . Let $G_u = G + E_u$ and $G_v = G + E_v$. If $(G; v) \prec_s (G; u)$, and $(G; w_i, v) \preceq_s (G; w_i, u)$ for each $i = 1, 2, \dots, r$, Then $SLEE(G_v) < SLEE(G_u)$.

For a vertex x and an edge e , let $SW_k(G; x, [e])$ be the set of all closed semi-edge walks of length k in the graph G starting at vertex x and containing the edge e .

Lemma 2.5. Let G be a graph and $H = G + e$, such that $e = uv \in E(\overline{G})$. If $(G; v) \preceq_s (G; u)$, then $(H; v) \preceq_s (H; u)$. Moreover, if $(G; v) \prec_s (G; u)$, then $(H; v) \prec_s (H; u)$.

Proof . We know that for each $z \in \{u, v\}$, and $k \geq 0$,

$$|SW_k(H; z)| = |SW_k(G; z)| + |SW_k(H; z, [e])|.$$

Since $(G; v) \preceq_s (G; u)$, $|SW_k(G; v)| \leq |SW_k(G; u)|$, for each $k \geq 0$. Thus there is a bijection $f_k : SW_k(G; v) \rightarrow A_k \subseteq SW_k(G; u)$, for each $k \geq 0$.

It is enough to show that $|SW_k(H; v, [e])| \leq |SW_k(H; u, [e])|$, for each $k \geq 0$. Let $W \in SW_k(H; v, [e])$. We can uniquely decompose W to $W = W_1 e W_2 e \dots e W_r$, such that $W_i \in SW_{k_i}(G; x, y)$, where $x, y \in \{u, v\}$, and $k_i \geq 0$, and $1 \leq i \leq r$. Note that W_i is a semi-edge walk in G and does not contain e , Thus the decomposition is unique. For each W_i exactly one of the following cases occurs:

- 1) $W_i \in SW_{k_i}(G; v, v)$. In this case we set $h(W_i) = f_{k_i}(W_i)$. Thus, $h(W_i) \in A_{k_i} \subseteq SW_{k_i}(G; u, u)$.
 - 2) $W_i \in A_{k_i} \subseteq SW_{k_i}(G; u, u)$. In this case, set $h(W_i) = f_{k_i}^{-1}(W_i) \in SW_{k_i}(G; v, v)$.
 - 3) $W_i \in SW_{k_i}(G; u, u) \setminus A_{k_i}$, or $W_i \in SW_{k_i}(G; u, v)$, or $W_i \in SW_{k_i}(G; v, u)$.
- In these cases, let h fix W_i , i.e. $h(W_i) = W_i$.

Now, it is easy to check that the map $h : SW_k(H; v, [e]) \rightarrow SW_k(H; u, [e])$ by the rule $h_k(W) = h_k(W_1 e W_2 e \dots e W_r) = h(W_1) e h(W_2) e \dots e h(W_r)$ is an injection.

Note that if there exists k_0 such that $|SW_{k_0}(G; v)| < |SW_{k_0}(G; u)|$, then f_{k_0} is not surjective. Thus h_{k_0} is not a surjection, and we have

$$|SW_{k_0}(H; v, [e])| < |SW_{k_0}(H; u, [e])|$$

which implies that $(H; v) \prec_s (H; u)$. \square

By a similar method, we prove the following statement:

Lemma 2.6. Let G be a graph and $H = G + e$, such that $e = uv \in E(\overline{G})$, and $(G; v) \preceq_s (G; u)$. If there exists a vertex $x \in V(G)$ such that $(G; x, v) \preceq_s (G; x, u)$, then $(H; x, v) \preceq_s (H; x, u)$. Moreover, if $(G; v) \prec_s (G; u)$ or $(G; x, v) \prec_s (G; x, u)$, then $(H; x, v) \prec_s (H; x, u)$.

Proof . Since $(G; v) \preceq_s (G; u)$, there is a bijection $f_k : SW_k(G; v) \rightarrow A_k \subseteq SW_k(G; u)$, for each $k \geq 0$. Similarly, since $(G; x, v) \preceq_s (G; x, u)$, there is a bijection $g_k : SW_k(G; x, v) \rightarrow B_k \subseteq SW_k(G; x, u)$, for each $k \geq 0$. It is obvious that for each $k \geq 0$,

$$|SW_k(H; x, z)| = |SW_k(G; x, z)| + |SW_k(H; x, z, [e])|$$

where $z \in \{v, u\}$. It is enough to show that for each $k \geq 0$,

$$|SW_k(H; x, v, [e])| \leq |SW_k(H; x, u, [e])|.$$

Let $W \in SW_k(H; x, v, [e])$. W decomposes uniquely to $W_1 e W_2 e \dots e W_r$, where W_i is a semi-edge walk of length k_i in G . Three cases will be considered as follows for W_1 :

- 1) If $W_1 \in SW_{k_1}(G; x, v)$, then we set $h_1(W_1) = g_{k_1}(W_1) \in B_{k_1} \subseteq SW_{k_1}(G; x, u)$.
- 2) If $W_1 \in B_{k_1} \subseteq SW_{k_1}(G; x, u)$, then set $h_1(W_1) = g_{k_1}^{-1}(W_1) \in SW_{k_1}(G; x, v)$.
- 3) If $W_1 \in SW_{k_1}(G; x, u) \setminus B_{k_1}$, then set $h_1(W_1) = W_1$.

If $1 < i \leq r$, then three cases will be considered as follows for W_i :

- 1) If $W_i \in SW_{k_i}(G; v)$, then we set $h_i(W_i) = f_{k_i}(W_i) \in A_{k_i}$.
- 2) If $W_i \in A_{k_i} \subseteq SW_{k_i}(G; u)$, then set $h_i(W_i) = f_{k_i}^{-1}(W_i) \in SW_{k_i}(G; v)$.
- 3) If $W_i \in SW_{k_i}(G; u) \setminus A_{k_i}$, or $W_i \in SW_{k_i}(G; v, u)$, or $W_i \in SW_{k_i}(G; u, v)$, then we set $h_i(W_i) = W_i$.

One can check easily that the map $h_k : SW_k(H; x, v, [e]) \rightarrow SW_k(H; x, u, [e])$ by the rule $h_k(W) = h_k(W_1 e W_2 e \dots e W_r) = h_1(W_1) e h_2(W_2) e \dots e h_r(W_r)$ is injective.

The second part of the lemma is clear. \square

3. The graph with maximum SLEE with given diameter

For $x \in V(G)$, the *eccentricity* $e(x)$ of x is the distance to a vertex of G farthest from x , i.e. $e(x) = \max\{d(x, y) : y \in V(G)\}$. The *diameter* $d(G)$ is the maximum eccentricity of the vertices, whereas the *radius* $r(G)$ is the minimum eccentricity. Also, x is a *central vertex* if $e(x) = r(G)$ and a *diametral path* is a shortest path between two vertices whose distance is equal to $d(G)$. For convenience, let us denote $\lceil \frac{d}{2} \rceil$ by \widehat{d} where is the smallest integer number greater than $\frac{d}{2}$.

It is obvious that K_n is the unique graph with diameter 1. Also, the path on n vertices P_n , is the unique graph with diameter $n - 1$. Furthermore, $K_n - e$ is the graph with maximum *SLEE* with diameter 2, where e is an edge of K_n .

Lemma 3.1. Let G be a graph with diameter d , and $P_{d+1} = v_0v_1 \dots v_d$ be a diametral path in G . If $d \geq 2$ and $x \in V(G) \setminus V(P_{d+1})$, then x has at most 3 neighbors in $V(P_{d+1})$.

Proof. Suppose that x has neighbors $v_{i_1}, v_{i_2}, \dots, v_{i_r}$ in P_{d+1} , where $r > 3$, and $i_1 < i_2 < \dots < i_r$. Since $i_r - i_1 > 2$, the path $P' = v_0v_1 \dots v_{i_1}xv_{i_r}v_{i_r+1} \dots v_d$ from v_0 to v_d is of length $d - i_r + i_1 + 2 < d$, which is a contradiction. \square

Let $n > 4$, and $2 < d < n - 1$, and $1 \leq j \leq \hat{d}$. We denote by $\mathcal{H}_{d,j}$, the set of all graphs $H_{d,j}$, constructed from K_{n-1-d} and $P_{d+1} = v_0v_1 \dots v_d$, by attaching each vertex of K_{n-d-1} to exactly 3 vertices of P_{d+1} , such that for each $x \in V(K_{n-d-1})$, there exists an index i , $\hat{d} - j \leq i \leq \hat{d} + j - 2$, where x is attached to v_i, v_{i+1} and v_{i+2} . Therefore, none of $v_i, 0 \leq i < \hat{d} - j$ or $\hat{d} + j < i \leq d$, has a neighbor in K_{n-d-1} . Note that $v_{\hat{d}}$ is a central vertex of the path P_{d+1} . For example, all graphs $H_{4,2}$ with $n = 7$ are shown in Fig.1.

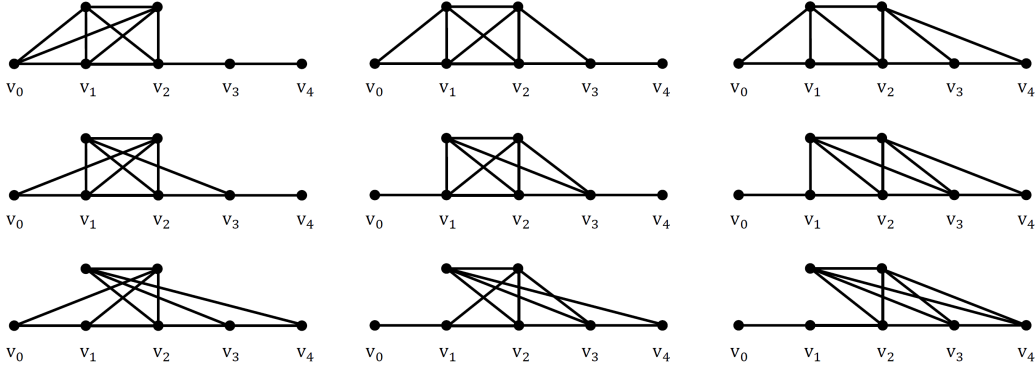


FIGURE 1. All graphs $H_{4,2}$ with $n = 7$.

Lemma 3.2. Let $n > 4$, and $2 < d < n - 1$, and $2 \leq j \leq \hat{d}$. If $H_j \in \mathcal{H}_{d,j}$, then either $H_j \in \mathcal{H}_{d,j-1}$, or there exists a graph, say $H_{j-1} \in \mathcal{H}_{d,j-1}$, such that $SLEE(H_j) < SLEE(H_{j-1})$.

Proof. Let $H_j \in \mathcal{H}_{d,j}$, and $N_K(v_i) = N(v_i) \cap V(K_{n-1-d})$, where $0 \leq i \leq d$ and $N(v_i)$ is the set of vertices that are adjacent to v_i . To facilitate the understanding of the proof, we divide the argument into two parts. We first discuss about $N_K(v_{\hat{d}-j})$ and then proceed to $N_K(v_{\hat{d}+j})$. Note that if d is odd, $j = 2$, and $N_K(v_{\hat{d}+2}) = \emptyset$, then by renaming the vertices of P_{d+1} such that v_i changes to v_{d-i} we conclude that $H_j \in \mathcal{H}_{d,j-1}$. Let $H_j \notin \mathcal{H}_{d,j-1}$. Therefore either at least one of the vertex subsets $N_K(v_{\hat{d}-j})$ or $N_K(v_{\hat{d}+j})$ is not empty, or d is odd and $j = 2$ and $N_K(v_{\hat{d}+2})$ is not empty.

If $N_K(v_{\hat{d}-j}) = \emptyset$, then we set $H'_{j-1} = H_j$. In this case, we have $SLEE(H'_{j-1}) = SLEE(H_j)$. Let $N_K(v_{\hat{d}-j})$ be not empty. For convenience, suppose that

$v = v_{\widehat{d}-j}$, $y = v_{\widehat{d}-j+1}$, $z = v_{\widehat{d}-j+2}$ and $u = v_{\widehat{d}-j+3}$. By the definition of $\mathcal{H}_{d,j}$, it is obvious that $N_K(v) \subseteq N_K(y) \subseteq N_K(z)$, and $N_K(v) \cap N_K(u) = \emptyset$.

Let $E = \{vx : x \in N_K(v)\}$, and $E' = \{ux : x \in N_K(v)\}$, and $H'_j = H_j - E$, and $H'_{j-1} = H'_j + E'$.

By lemma 2.4, to show that $SLEE(H_j) < SLEE(H'_{j-1})$, it is enough to prove the following statements:

- 1) $(H'_j; v) \prec_s (H'_j; u)$.
- 2) $(H'_j; x, v) \preceq_s (H'_j; x, u)$, for each $x \in N_K(v)$.

We start the prove of (1) by the following claim:

Claim. $(H'_j; y) \preceq_s (H'_j; z)$:

To prove the claim, let $W \in SW_k(H'_j - e; y)$, where $e = yz$, and $k \geq 0$. We can decompose W to $W = W_1 W_2 W_3$, where W_1 and W_3 are as long as possible and consisting of just the vertices v_0, v_1, \dots, y , and edges in $\{v_t v_{t+1} : 0 \leq t \leq \widehat{d} - j\} \cup \{yx : x \in N_K(y)\}$, and $W_2 \in SW_{k_2}(H'_j - e; x, w)$, where $x, w \in N_K(y) \subseteq N_K(z)$. Suppose that W'_i obtains from W_i , for $i = 1, 3$, by replacing each vertex v_t by v_a , and each edge $v_t v_{t+1}$ by $v_a v_{a-1}$, and each edge yx by zx , where $x \in N_K(y)$, and $a = 2\widehat{d} - 2j - t + 3$ (In fact, the distance between v_t and y is equal to the distance between v_a and z in P_{d+1}).

It is easy to check that the map $f'_k : SW_k(H'_j - e; y) \rightarrow SW_k(H'_j - e; z)$ defining by the rule $f'_k(W_1 W_2 W_3) = W'_1 W_2 W'_3$ is injective. Thus $(H'_j - e; y) \preceq_s (H'_j - e; z)$. Now, the claim follows from lemma 2.5.

Let $f_k : SW_k(H'_j; y) \rightarrow SW_k(H'_j; z)$ be an injection, for each $k \geq 0$. If $W \in SW_k(H'_j; v)$, then W can be decomposed to $W = W_1 W_2 W_3$, where $W_2 \in SW_{k_2}(H'_j; y)$ is as long as possible. Let W'_i obtain from W_i , for each $i = 1, 3$, by replacing each vertex v_t by v_a , and each edge $v_t v_{t+1}$ by $v_a v_{a-1}$, where $a = 2\widehat{d} - 2j - t + 3$. The map $g_k : SW_k(H'_j; v) \rightarrow SW_k(H'_j; u)$, defining by the rule $g_k(W_1 W_2 W_3) = W'_1 f_{k_2}(W_{k_2}) W'_3$ is injective. Note that if $j > 2$ or d is even, then the path $v_0 v_1 \dots v$ is a proper subgraph of the path $v_d v_{d-1} \dots u$. Also, if d is odd and $j = 2$, then $N_K(u) \neq \emptyset$, implies that $\deg_{H'_j}(v) = 2 < \deg_{H'_j}(u)$. Thus $(H'_j; v) \prec_s (H'_j; u)$ which is (1).

By a similar method used above, we prove the statement (2). First, we claim that:

Claim. $(H'_j; x, y) \preceq_s (H'_j; x, z)$, for each $x \in N_K(v)$.

To prove the claim, let $x \in N_K(v)$, and $W \in SW_k(H'_j - e; x, y)$ where $e = yz$. We can decompose W to $W = W_1 W_2$ such that $W_1 \in SW_{k_1}(H'_j - e; x, w)$ is as long as possible, where $w \in N_K(y)$, and $W_2 \in SW_{k_2}(H'_j - e; w, y)$. Suppose that W'_2 obtains from W_2 by replacing each vertex v_t by v_a , and the edge wy by wz , and each edge $v_t v_{t+1}$ by $v_a v_{a-1}$, where $a = 2\widehat{d} - 2j - t + 3$.

One can easily check that the map $h'_k : SW_k(H'_j - e; x, y) \rightarrow SW_k(H'_j - e; x, z)$ defining by the rule $h'_k(W_1 W_2) = W_1 W'_2$ is injective. Thus $(H'_j - e; x, y) \preceq_s (H'_j - e; x, z)$. Now, the claim follows from lemma 2.6.

Consider $h_k : SW_k(H'_j; x, y) \rightarrow SW_k(H'_j; x, z)$ is an injective map, for each $k \geq 0$. Let $W \in SW_k(H'_j; x, v)$. we can decompose W to $W = W_1 W_2$, where $W_1 \in SW_{k_1}(H'_j; x, y)$ is as long as possible, and $W_2 \in SW_{k_2}(H'_j; y, v)$. Let W'_2 obtain from W_2 by replacing each vertex v_t by v_a , and replacing each edge $v_t v_{t+1}$ by $v_a v_{a-1}$, where $a = 2\hat{d} - 2j - t + 3$. It is elementary to show that the map $l_k : SW_k(H'_j; x, v) \rightarrow SW_k(H'_j; x, u)$ defining by the rule $l_k(W_1 W_2) = h_{k_1}(W_1) W'_2$ is an injection. Thus, $(H'_j; x, v) \preceq_s (H'_j; x, u)$ for each $x \in N_K(v)$. It follows the statement (2).

Now, by the above discussion and lemma 2.4, we have $SLEE(H_j) \leq SLEE(H'_{j-1})$, with equality if and only if $H'_{j-1} = H_j$. The first part of the argument ends here.

If $N_K(v_{\hat{d}+j})$ is empty or d is odd and $j = 2$, then $H'_{j-1} \in \mathcal{H}_{d,j-1}$. In this case, set $H_{j-1} = H'_{j-1}$, and of course $SLEE(H_{j-1}) = SLEE(H'_{j-1})$. Let $H'_{j-1} \notin \mathcal{H}_{d,j-1}$, Then $N_K(v_{\hat{d}+j})$ is not empty. By repeating the above discussion for $v = v_{\hat{d}+j}$, $y = v_{\hat{d}+j-1}$, $z = v_{\hat{d}+j-2}$ and $u = v_{\hat{d}+j-3}$, we get the graph $H_{j-1} = H'_{j-1} - E + E'$, such that $H_{j-1} \in \mathcal{H}_{d,j-1}$ and $SLEE(H'_{j-1}) < SLEE(H_{j-1})$. Therefore,

$$SLEE(H_j) \leq SLEE(H'_{j-1}) \leq SLEE(H_{j-1}) \in \mathcal{H}_{d,j-1}$$

with equalities hold, if and only if graphs are equal. \square

The following theorem is our main result of this section, which is determined the unique graph with maximum $SLEE$ among the set of all unicyclic graphs with diameter d , where $2 < d < n - 1$.

Theorem 3.3. Let $2 < d < n - 1$. If G has maximum $SLEE$ with diameter d , then $G = H_{d,1}$.

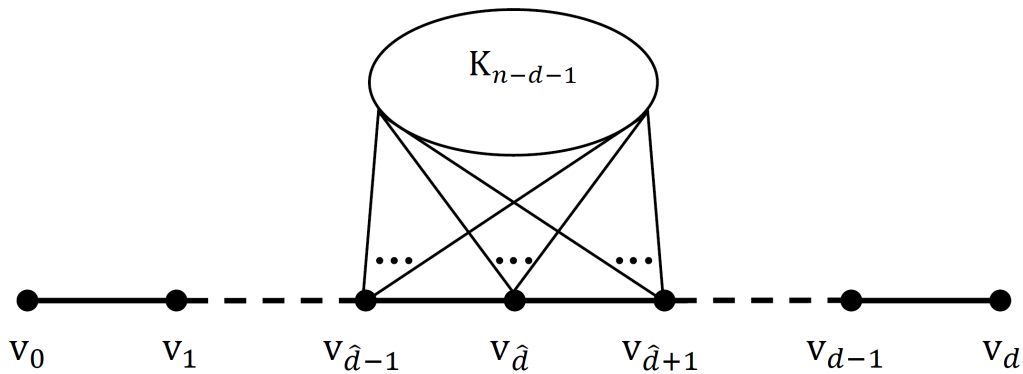


FIGURE 2. The unique graph on n vertices having the maximum $SLEE$ with diameter d .

Proof . Suppose that G is a graph, having maximum $SLEE$ with diameter d . Let $P_{d+1} = v_0v_1 \dots v_d$ be a diametrical path in G , and H be the graph obtained from G by adding some edges such that:

- (a) x is adjacent with exactly 3 vertices of P_{d+1} in H , say v_i, v_{i+1} and v_{i+2} , for each $x \in V(G) \setminus V(P_{d+1})$.
- (b) $H - V(P_{d+1})$ is a complete graph on $n - 1 - d$ vertices.

By lemma 3.1, such a graph H exists. Obviously, we have $H \in \mathcal{H}_{d,j}$, for some j , $1 \leq j \leq \widehat{d}$, and $SLEE(G) \leq SLEE(H)$, with equality if and only if $G = H$.

If $j > 1$, then by lemma 3.2, we may get a sequence of graphs, say $H_{d,j-1}, H_{d,j-2}, \dots, H_{d,1}$, such that for each t , $H_{d,t} \in \mathcal{H}_{d,t}$, and

$$SLEE(G) \leq SLEE(H) \leq SLEE(H_{d,j-1}) \leq SLEE(H_{d,j-2}) \leq \dots \leq SLEE(H_{d,1})$$

with equalities hold, if and only if the graphs are equal. since the diameter of $H_{d,1}$ is d , and G has the maximal $SLEE$ among the set of all graphs with diameter d , hence $SLEE(G) = SLEE(H_{d,1})$ which implies that $G = H_{d,1}$, as expected. \square

4. The graph with maximum SLEE with given number of cut vertices

A *cut vertex* of a graph is a vertex whose removal increases the number of components of the graph. Let G be a connected graph and x be a vertex of G , a *block* of G is defined to be a maximal subgraph without cut vertices.

A *pendent path* at x in a graph G is a path in which no vertex other than x is incident with any edge of G outside the path, where $\deg_G(x) \geq 3$. In particular, we consider a vertex x as a pendent path at x of length zero in G only when x is neither a pendent vertex nor a cut vertex of G . Let G and H be two vertex-disjoint connected graphs, such that $x \in V(G)$ and $y \in V(H)$. We denote the *coalescence* of G and H by $G(x) \circ H(y)$, which is obtained by identifying the vertex x of G with the vertex y of H .

Lemma 4.1. Let H_1 and H_2 be two graphs and $P_s = y_0y_1 \dots y_{s-1}$ be a path on s vertices, and $u \in V(H_2)$ and $xy \in E(H_1)$, such that $x \neq y$. Let $G = (H_1(y) \circ P_s(y_0))(x) \circ H_2(u)$. If H_2 contains a path $Q_{s+2} = ux_1x_2 \dots x_{s+1}$, then $SLEE(G) < SLEE(G - E_y + E_{x_1})$, where $E_y = \{yw : w \in N_{H_1}(y) \setminus \{x\}\}$, $E_{x_1} = \{x_1w : w \in N_{H_1}(y) \setminus \{x\}\}$ and $N_{H_1}(y)$ is the set of vertices of H_1 that are adjacent to y .

Proof . Let $G' = G - E_y$. By lemma 2.4, it is enough to show that $(G'; y) \prec_s (G'; x_1)$, and $(G'; w, y) \preceq_s (G'; w, x_1)$, for each $w \in N_{H_1}(y) \setminus \{x\}$. Let $P'_{s+1} = xy_0y_1 \dots y_{s-1}$, and $A_k = SW_k(G'; y) \setminus SW_k(P'_{s+1}; y)$, and $B_k = SW_k(G'; x_1) \setminus SW_k(Q_{s+2}; x_1)$. Since P'_{s+1} is a proper subgraph of Q_{s+2} , it is easy to show that $|SW_k(P'_{s+1}; y)| \leq |SW_k(Q_{s+2}; x_1)|$, and for some $k = k_0 \geq s$, inequality is strict.

Let $W \in A_{k_0}$. We may decompose W to $W_1W_2W_3$ such that $W_2 \in SW_{k_2}(G'; x)$

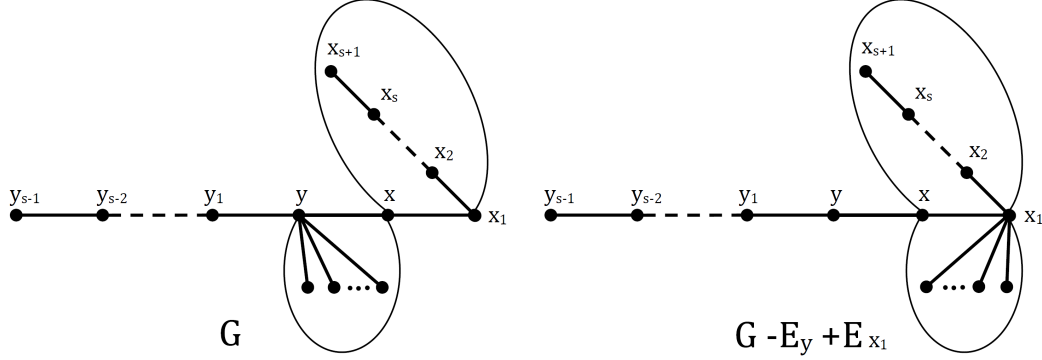


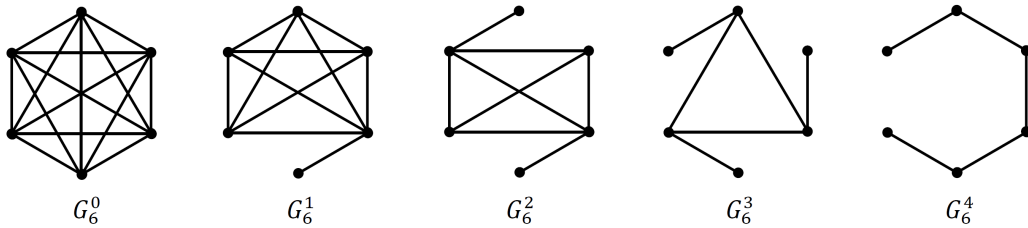
FIGURE 3. An illustration of graphs in Lemma 4.1 .

is as long as possible and $W_1 \in SW_{k_1}(G'; y, x)$, $W_3 \in SW_{k_3}(G'; x, y)$ and $k = k_1 + k_2 + k_3$. Let W'_j obtain from W_j by replacing each y_i by x_{i+1} , where $j = 1, 3$ and $i = 0, 1, \dots, s-1$. The map $f : A_k \rightarrow B_k$ defined by the rule $f(W_1 W_2 W_3) = W'_1 W_2 W'_3$ is injective. Thus $|A_k| \leq |B_k|$. Therefore $|SW_k(G'; y)| \leq |SW_k(G'; x_1)|$, and for some $k = k_0$ the inequality is strict. Hence $(G'; y) \prec_s (G'; x_1)$.

Let $w \in N_{H_1}(y) \setminus \{x\}$, and $W \in SW_k(G'; w, y)$. We can decompose W uniquely to $W_1 W_2$, such that $W_1 \in SW_{k_1}(G'; w, x)$ is as long as possible. Let W'_2 obtain from W_2 by replacing each y_i by x_{i+1} , where $W_2 \in SW_{k_2}(G'; x, y)$, $k = k_1 + k_2$ and $i = 0, 1, \dots, s-1$.

The map $g_{w,k} : SW_k(G'; w, y) \rightarrow SW_k(G'; w, x_1)$ defining by the rule $g_{w,k}(W_1 W_2) = W_1 W'_2$ is injective. Thus $|SW_k(G'; w, y)| \leq |SW_k(G'; w, x_1)|$ for each k . Therefore $(G'; w, y) \preceq_s (G'; w, x_1)$, for each $w \in N_{H_1}(y) \setminus \{x\}$. \square

Let $0 \leq r \leq n-2$. Suppose that G_n^r is the graph obtained from K_{n-r} by attaching $n-r$ pendent path of orders n_1, n_2, \dots, n_{n-r} to its vertices, where each vertex of K_{n-r} has exactly one pendent path and $|n_i - n_j| \leq 1$ for $1 \leq i, j \leq n-r$. More precisely, each pendent path is of order $\lfloor \frac{r}{n-r} \rfloor$ or $\lfloor \frac{r}{n-r} \rfloor + 1$. For example, the graphs G_6^r with $r = 0, 1, 2, 3, 4$ are shown in Fig.4 .

FIGURE 4. The graphs G_6^r with $r = 0, 1, 2, 3, 4$.

Theorem 4.2. If $0 \leq r \leq n - 2$, then G_n^r is the unique graph with maximum *SLEE* among all graphs on n vertices with r cut vertices.

Proof . Since $P_n = G_n^{n-2}$ is the unique graph with $n - 2$ cut vertices, the case $r = n - 2$ is obvious. If $r = 0$, then by lemma 2.3, $K_n = G_n^0$ is the unique graph on n vertices with maximum *SLEE*. Let $1 \leq r \leq n - 3$, and G be a graph with maximum *SLEE* among all graphs on n vertices with r cut vertices.

First, we prove that G is connected. Otherwise, if G is not connected and x is a cut vertex of G , then x is also a cut vertex of a component, say G_1 of G . Let G_2 be another component of G . If G_2 has a cut vertex, say y , then set $G' = G + \{xy\}$. If G_2 has no cut vertex, then suppose that G' is the graph obtained from G by attaching x to each vertex of G_2 . It is easy to check that in both cases, G' is a graph with r cut vertex and $SLEE(G) < SLEE(G')$, a contradiction. Thus G is connected.

By lemma 2.3, every block of G is complete. Let x be a cut vertex contained in at least 3 blocks, say B_1 , B_2 and B_3 . Suppose that, B_1 and B_3 will be disjointed if the vertex x is removed. Let G' be the graph obtained from G by attaching each vertex of B_1 to each vertex of B_2 . Obviously, G' has r cut vertex and by lemma 2.3, $SLEE(G) < SLEE(G')$, a contradiction. Thus, each cut vertex of G is contained in exactly two blocks.

Suppose that G has at least one block with at least 3 vertices. Otherwise, since each block of G has 2 vertices, G is a tree with maximum degree 2. Thus $G \cong P_n$, and $r = n - 2$, a contradiction.

Let P_s be a pendent path with minimum length in G at x . Obviously, x lies in a block of G , say B , with at least 3 vertices. Note that if $s = 1$, then x is not a cut vertex.

For each $y \in V(B)$, let H_y be the component of $G - E(B)$ which is containing y . Obviously, $H_x = P_s$. Let $y \in V(B)$ such that $y \neq x$. Let H be the component of $G - (E(H_x) \cup E(H_y))$ containing y . We have $G \cong (H(x) \circ H_x(x))(y) \circ H_y(y)$. Suppose that H_y is not a path. Since P_s has minimal length, there is a pendent path on at least s vertices at a vertex in H_y , say z , where $z \neq y$. Thus H_y contains a path on at least $s + 2$ vertices with an end vertex y . Note that since H_y is not a path, we can choose some vertices of H_y and construct the path of length at least $s + 2$ with an end vertex y . By lemma 4.1, we may get another graph on n vertices with r cut vertices, which has larger *SLEE*, a contradiction. Therefore, H_y is a pendent path, say P_t at y .

By the choice of P_s , we have $t \geq s$. If $t \geq s + 2$, then by lemma 4.1, we may obtain another graph on n vertices with r cut vertices, which has a larger *SLEE* than G , a contradiction. Therefore, for each $y \in V(B)$, $H_y \cong P_s$ or P_{s+1} . Hence $G \cong G_n^r$. \square

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